

Financial mathematics

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Definition 1. The *present value* p_0 or the *principal* is the amount initially borrowed or invested. The *future value* p_t is the principal after a period of time t .

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Definition 2. Interest rates expressed per annum are called *nominal rates*, i . The *annual percentage rate* or *effective annual rate* i_a is the equivalent annual rate of different interest rates variously compounded.

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Definition 3. A *sequence* is a list of numbers which follows a definite pattern. It is called an *arithmetic sequence* if each of its terms is obtained from the term immediately preceding it by an addition of a constant d , which is called the *common difference*. It is called a *geometric sequence* if each of its term is obtained from the previous term by a multiplication of a constant r , the *common ratio*.

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Definition 4. A *series* is the sum of the terms of sequence. It is called a *finite series* is one whose number of terms is finite, otherwise it is called an *infinite series*. An *arithmetic series* or *arithmetic progression* is the sum of the terms of an arithmetic sequence. Likewise a *geometric series* or *geometric progression* is the sum of the terms of geometric sequence.

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Theorem 1. The value of the n^{th} term of an arithmetic series is

$$T_n = a + (n - 1)d$$

The sum of its first n terms is

$$S_n = \frac{n}{2} (2a + (n - 1)d)$$

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Problem 1. Prove Theorem 1.

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Theorem 2. The n^{th} term of a geometric series is

$$T_n = ar^{n-1}$$

The sum of the first n terms of it is

$$\begin{aligned} S_n &= a + ar + \cdots + ar^{n-1} \\ &= \frac{a(1 - r^n)}{1 - r} \\ &= \frac{a(r^n - 1)}{r - 1} \end{aligned}$$

When the number of terms approaches infinity, the summation in cases where $r < 1$ becomes

$$S_\infty = \frac{a}{1 - r}$$

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Definition 5. A *simple interest* is a fixed percentage of the principal paid to an investor each year. A *compound interest* is an interest paid on the principal plus any interest accumulated in previous years.

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Theorem 3. The present value for simple interest is

$$p_t = p_0(1 + it)$$

where i is the interest rate and t the time in years.

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Problem 2. Prove Theorem 3

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Theorem 4. The present value in the case of compound interest is

$$p_t = p_0(1 + i)^t$$

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Note 1. The interest may be compounded more than once a year, for example biannually, quarterly, monthly, weekly, daily, or continuously. Each time period is called a *conversion period* or *interest period*. The interest rate applied at each conversion is i/m , where m is the number of conversion periods per year. The number of conversion periods over t years is then $n = mt$.

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Theorem 5. The present value at the end of n conversion periods is

$$p_t = p_0 \left(1 + \frac{i}{m}\right)^n = p_0 \left(1 + \frac{i}{m}\right)^{mt}$$

where all the variables and parameters are as previously defined.

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Problem 3. Prove Theorem 5.

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Theorem 6. When the number of compoundings per year becomes very large, the present value becomes

$$p_t = p_0 e^{it}$$

Proof. Since $p_t = p_0 \left(1 + \frac{i}{m}\right)^{mt}$ and $\lim_{m \rightarrow \infty} \left(1 + \frac{i}{m}\right)^m = e^i$, we have the proof. ¶

Theorem 7. The annual percentage rate when compounding occurs m times per year is

$$i_a = \left(1 + \frac{i}{m}\right)^m - 1$$

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Problem 4. Prove Theorem 7.

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Bibliography

Teresa Bradley. *Essential mathematics for economics and business*. 2nd ed. 2002